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# Multigroup half space moment approximations to the radiative heat transfer equations ☆

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**Abstract**

The half space moment approximation to the radiative heat transfer equations developed in [J. Comp. Phys. 180 (2002) 584] for frequency-independent absorption and scattering is extended to frequency-dependent coefficients using averaging over frequency groups. We compare numerical results obtained with this new model to known approximations in different physical regimes.


*Keywords:* Radiative transfer equations; Moment methods; Entropy minimization; Half fluxes; Multigroup equations

## 1. Introduction

In recent years, the interest in numerically tractable approximations to the radiative heat transfer equations has drastically increased. Applications range from industrial cooling processes (e.g., glass cooling) over astrophysics to combustion (e.g., in gas turbine combustion chambers). Since radiative heat transfer (RHT) often plays a role in complex physical situations involving for example fluid flow and chemical reactions, one is interested in substituting the system of integro-differential equations describing RHT by a mathematically less complicated, yet accurate, approximate model. Examples of such approximate models are diffusion approximations [7], higher order diffusion approximations like $P_N$ and $SP_N$ equations (cf. [6,8] and references therein), and moment models [2,9,11,12].

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The model that we present in this paper is a further development of the model introduced in [5]. By using half space moments, i.e., by averaging the radiative intensity over the directions going to the left and to the right separately, one can capture highly anisotropic physical situations. In numerical experiments the grey half space moment model has shown to be very accurate at a low computational cost [4,5]. The term "grey" comes from the astrophysics literature and means "frequency-independent". In this paper, we extend the grey half space moment model to a multigroup half space moment model taking into account frequency-dependent absorption and scattering coefficients.

We consider the RHT equations in slab geometry

$$\rho_m c_m \partial_t T - \partial_x(k\partial_x T) = 2\pi \int_{\nu_0}^{\infty} \int_{-1}^{1} \kappa(I_\nu - B_\nu(T))\,\mathrm{d}\mu\,\mathrm{d}\nu, \tag{1.1}$$

$$\forall \nu > \nu_0 : \frac{1}{c}\partial_t I_\nu + \mu\partial_x I_\nu = \sigma\left(\frac{1}{2}\int_{-1}^{1} I_\nu \mathrm{d}\mu - I_\nu\right) + \kappa(B_\nu(T) - I_\nu). \tag{1.2}$$

In these equations, $I_\nu(x,t,\mu)$ denotes the radiative intensity at point $x \in ]0,1[$, time $t$, frequency $\nu$, travelling into direction $\mu \in [-1,1]$. Furthermore, $T(x,t)$ is the material temperature. The heat conductivity is denoted by $k$. In the opaque frequency interval $[0,\nu_0]$, radiation is immediately absorbed, i.e., the medium behaves like a blackbody. In this frequency range, $I_\nu \equiv B_\nu(T)$. Formally, this corresponds to $\kappa \to \infty$. The absorption coefficient $\kappa$ and the scattering coefficient $\sigma$ generally depend on the frequency. Here, $B_\nu$ is the Planck equilibrium distribution

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{\exp(\frac{h\nu}{kT}) - 1}. \tag{1.3}$$

We supplement this system with the following boundary conditions. For the material temperature we consider the heat flux through the boundary due to advection and emission/absorption:

$$k\frac{\partial T}{\partial n} = h(T_\mathrm{b} - T) + \alpha\pi\int_0^{\nu_0}(B_\nu(T_\mathrm{b}) - B_\nu(T))\,\mathrm{d}\nu. \tag{1.4}$$

Here, $T_\mathrm{b}$ is the outside temperature and $\alpha$ is the hemispheric emmisivity/absorptivity. Furthermore, we prescribe semi-transparent boundary conditions for the ingoing radiation

$$I_\nu(\mu) = \rho(-\mu)I_\nu(-\mu) + (1 - \rho(\mu))B_\nu(T_\mathrm{b}). \tag{1.5}$$

The reflectivity $\rho$ can be computed using Snell's law. Finally, we use suitable initial values for $T$ and $I_\nu$.

## 2. Half space moment closure

Moment models are obtained by testing (1.2) with functions depending on direction, in our case $(1,\mu)^T$, then integrating the result over all the directions and frequencies. Then, the system does only depend on time and space variables, and is hence far cheaper to solve. However, this has a cost since we are not always able to reproduce neither frequency dependent problems nor very stiff directional configurations such as the collision of two opposite beams [1,5].

In order to solve this difficulty, we do not average over all directions and all frequencies but distinguish photons going to the left and to the right and different frequency bands.

Let

$$\langle g\rangle_m^+ = \frac{1}{c}\int_{\nu_{m-\frac{1}{2}}}^{\nu_{m+\frac{1}{2}}}\int_0^1 g\,\mathrm{d}\mu\,\mathrm{d}\nu \quad \text{and} \quad \langle g\rangle_m^- = \frac{1}{c}\int_{\nu_{m-\frac{1}{2}}}^{\nu_{m+\frac{1}{2}}}\int_{-1}^0 g\,\mathrm{d}\mu\,\mathrm{d}\nu, \tag{2.1}$$

denote the average over all right/left-going photons in the $m$th frequency band $[\nu_{m-\frac{1}{2}}, \nu_{m+\frac{1}{2}}[$. We denote the bands as half-open intervals to have mathematically disjoint sets. However, since only integrals over the bands matter, one could also use closed intervals. The moments $E_{R,m}^+ = \langle I_\nu\rangle_m^+$, $F_{R,m}^+ = \langle \Omega I_\nu\rangle_m^+$ and $P_{R,m}^+ = \langle(\Omega\otimes\Omega)I_\nu\rangle_m^+$ are respectively, the radiative energy, the radiative flux and the radiative pressure inside the $m$th group and the positive half-space. The quantities for the negative half space are defined in analogy.

Testing (1.2) with $(1,\mu)^T$ and averaging with the above defined averages we get

$$\partial_t E_{R,m}^+ + \partial_x F_{R,m}^+ = c\hat{\kappa}_m^+ a\theta_{m,+}^4 - \tilde{\kappa}_m^+ E_{R,m}^+ + \tilde{\sigma}_m^+\left(\frac{E_{R,m}^+ + E_{R,m}^-}{2} - E_{R,m}^+\right), \tag{2.2}$$

$$\frac{1}{c}\partial_t F_{R,m}^+ + c\partial_x P_{R,m}^+ = c\hat{\kappa}_m^+\frac{a}{2}\theta_{m,+}^4 - \breve{\kappa}_m^+ F_{R,m}^+ - \tilde{\sigma}_m^+\left(\frac{E_{R,m}^+ + E_{R,m}^-}{4} - F_{R,m}^+\right), \tag{2.3}$$

and

$$\partial_t E_{R,m}^- + \partial_x F_{R,m}^- = c\hat{\kappa}_m^- a\theta_{m,-}^4 - \tilde{\kappa}_m^- E_{R,m}^- + \tilde{\sigma}_m^-\left(\frac{E_{R,m}^+ + E_{R,m}^-}{2} - E_{R,m}^-\right), \tag{2.4}$$

$$\frac{1}{c}\partial_t F_{R,m}^- + c\partial_x P_{R,m}^- = -c\hat{\kappa}_m^-\frac{a}{2}\theta_{m,-}^4 - \breve{\kappa}_m^- F_{R,m}^- - \tilde{\sigma}_m^-\left(-\frac{E_{R,m}^+ + E_{R,m}^-}{4} - F_{R,m}^-\right), \tag{2.5}$$

where we have used the following frequency averages of the frequency dependent quantities $\kappa$ and $\sigma$:

$$\hat{\kappa}_m^+ = \frac{\langle\kappa B_\nu\rangle_m^+}{\langle B_\nu\rangle_m^+}, \quad \tilde{\kappa}_m^+ = \frac{\langle\kappa I_\nu\rangle_m^+}{\langle I_\nu\rangle_m^+}, \quad \breve{\kappa}_m^+ = \frac{\langle\kappa\mu I_\nu\rangle_m^+}{\langle\mu I_\nu\rangle_m^+} \quad \text{and} \quad \tilde{\sigma}_m^+ = \frac{\langle\sigma I_\nu\rangle_m^+}{\langle I_\nu\rangle_m^+}. \tag{2.6}$$

### 2.1. Entropy minimization

For each $m$ (2.2)–(2.5) is a system of four equations for six unknown moments. To obtain a well-posed system one usually expresses the highest moment, here $P_{R,m}^\pm$, as a function of the lower order moments, here $E_{R,m}^\pm$ and $F_{R,m}^\pm$. This is referred to as "closure" of the system.

To close the system here, we use entropy minimization, see [2,3,10,11]. Compare [5] for the grey half space model and [13] for the multigroup full space model.

Let us first recall the definition of the radiative entropy,

$$h_R(I) = \frac{2k\nu^2}{c^3}[n_I\ln n_I - (n_I+1)\ln(n_I+1)], \tag{2.7}$$

where

$$n_I = \frac{c^2}{2h\nu^3}I. \tag{2.8}$$

According to the entropy minimization principle, we determine a distribution function that minimizes the radiative entropy under the constraint that it reproduces the lower order moments

$$H_R(\mathscr{I}) = \min_I \left\{ H(I) = \sum_m (\langle h_R(I)\rangle_m^+ + \langle h_R(I)\rangle_m^-) | \right. \\ \left. \forall m : \langle I\rangle_m^\pm = E_m^\pm \quad \text{and} \quad c\langle \mu I\rangle_m^\pm = F_m^\pm \right\}. \tag{2.9}$$

This gives the closure function

$$\mathscr{I}(\Omega, \nu) = \sum_m 1_{[\nu_{m-\frac{1}{2}};\nu_{m+\frac{1}{2}}[} \frac{2h\nu^3}{c^2} \left[ \exp\frac{h\nu}{k} \left( \alpha_m^+(1+\beta_m^+\mu^+) + \alpha_m^-(1+\beta_m^-\mu^-)\right) - 1 \right]^{-1}, \tag{2.10}$$

where $\alpha_m^\pm, \beta_m^\pm$ are Lagrange multipliers, that are defined to reproduce the moments of $\mathscr{I}$.

### 2.2. Inversion of the system

The next step is to express the Lagrange multipliers $\alpha_m^\pm, \beta_m^\pm$ as functions of $E_{R,m}^\pm$, $F_{R,m}^\pm$ and to substitute

$$P_{R,m}^\pm \approx \left\langle \mathscr{I}(\alpha_m^\pm, \beta_m^\pm)\right\rangle_m^\pm = \left\langle \mathscr{I}(E_{R,m}^\pm, F_{R,m}^\pm)\right\rangle_m^\pm. \tag{2.11}$$

Hence we obtain a system for $E_{R,m}^\pm$ and $F_{R,m}^\pm$.

For the grey half space model [5], the Lagrange multipliers as functions of the moments can be computed explicitly. However, with the introduction of multigroup variables this is not the case anymore. Integrations require the knowledge of the following function:

$$\Xi(\eta) = \int_0^\eta \xi^3[\exp(\xi) - 1]^{-1}\,\mathrm{d}\xi. \tag{2.12}$$

For example

$$E_m^+ = \frac{1}{c}\int_0^1 \int_{\nu_{m-\frac{1}{2}}}^{\nu_{m+\frac{1}{2}}} \mathscr{I}\,\mathrm{d}\nu\,\mathrm{d}\mu = \int_0^1 \frac{2k^4}{h^3c^3}(\alpha_m^+(1+\beta_m^+\mu^+))^{-1}\left(\Xi\left(\nu'_{m+\frac{1}{2}}\right) - \Xi\left(\nu'_{m-\frac{1}{2}}\right)\right)\mathrm{d}\mu, \tag{2.13}$$

with $\nu' = \frac{h\nu}{k}\alpha^+(1+\beta^+\mu^+)$. Unfortunately, except for $\eta = 0$ and $\eta = +\infty$ there is no analytic expression of $\Xi$. A numerical calculation would be too expensive since we have to be very accurate. Therefore, in [13] an approximation was introduced, which can also be used here,

$$\Xi(\eta) \simeq C_\infty + \exp(-C_*\eta)\sum_{i=0}^{i\max} C_i\eta^i. \tag{2.14}$$

The constants $C_i$ are chosen so that the approximation has a very good behaviour in the vicinity of $\eta = 0$. For our applications, taking $i\max = 5$ is sufficient.

Once this approximation is made, it is possible to integrate and hence to compute the Lagrange multipliers of the minimization problem as functions of the moments. Then, we are able to compute the radiative pressures as functions of the radiative energies and fluxes. Moreover, we can show that we can write the pressures in Eddington form, $P_R^\pm = D_R^\pm E_R^\pm$, where

$$D_m^\pm = \frac{(1-\chi_m^\pm)}{2} I_d + \frac{(3\chi_m^\pm - 1)}{2}\frac{F_m^\pm \otimes F_m^\pm}{\|F_m^\pm\|^2}. \tag{2.15}$$

The scalars $\chi_m^\pm$ are called Eddington factors.

### 2.3. Properties and numerical schemes

The multigroup half space model keeps the interesting properties of the other moment models closed by entropy minimization, that is to say

- The main physical properties remain: conservation of the total energy and dissipation of the total entropy. Moreover, the addition of multigroup allows to have a better balanced-energy in the case of strongly frequency-dependant problems.
- The model naturally limits the flux. This property can be expressed as follows:

$$\forall m, \ \frac{F_m^{\pm}}{cE_m^{\pm}} < 1. \tag{2.16}$$

  This means that the photons cannot travel faster than the speed of the light. We note that this important property is often not satisfied by macroscopic models.
- For 1D problems, it is very easy to make a simple numerical scheme that can efficiently solve every possible angular configuration. This is done only by using upwind schemes (see [5]). We chose to develop only a four-moments model to obtain a simple and very competitive model. However, in some situations one might need more moments to capture the physical solution [12].
- The cost of the method is low and can be lowered to be less than the number of groups times the cost of the half space model by doing a pressure precalculation.

These properties are the most important ones but it is to note that the multigroup half space model keeps all the properties (and limitations) of both the half space [5] and multigroup full space [13] models.

## 3. Numerical results

First we consider only the equation for the radiative intensity with a fixed matter temperature profile. We divide the spectrum into four bands $[\lambda_{i-\frac{1}{2}}, \lambda_{i+\frac{1}{2}}[$ ($]\nu_{i+\frac{1}{2}}, \nu_{i-\frac{1}{2}}]$, respectively) with piecewise constant $\kappa_i$ on $[\lambda_{i-\frac{1}{2}}, \lambda_{i+\frac{1}{2}}[$. We used $\lambda_{\frac{1}{2}} = 0$ μm, $\lambda_{\frac{3}{2}} = 1.035$ μm, $\lambda_{\frac{5}{2}} = 2.07$ μm, $\lambda_{\frac{7}{2}} = 7$ μm and $\lambda_{\frac{9}{2}} = \infty$ and $\sigma = 0$.

In Figs. 1–3 we compare the results obtained with the half space moment model to the solution of the full RHT equations using a source iteration as well as diffusive $P_1$ and $SP_3$ approximations. For details on these equations we refer the reader to [8]. The classical Rosseland approximation gives in all cases considered here far less accurate results.

For the radiative energy

$$E_R = \int_0^{\infty} \int_{-1}^{1} I_\nu \, \mathrm{d}\mu \, \mathrm{d}\nu = \sum_m (E_{R,m}^+ + E_{R,m}^-), \tag{3.1}$$

we define, in analogy to Stefan's law, the radiative temperature

$$T_R := \left( \frac{2\pi E_R}{a} \right)^{1/4}. \tag{3.2}$$

The parameters corresponding to Fig. 1 are $\kappa_1 = 100 \ \mathrm{m}^{-1}$, $\kappa_2 = 1 \ \mathrm{m}^{-1}$, $\kappa_3 = 10 \ \mathrm{m}^{-1}$, $\kappa_4 = \infty$ and represent a rather diffusive, optically thick physical regime. The half space model performs better than the diffusive approximations which are designed for this physical situation. The differences become more striking in Fig. 2 where we chose a rather opposite physical regime with large photon mean free path, $\kappa_1 = 0.1 \ \mathrm{m}^{-1}$,

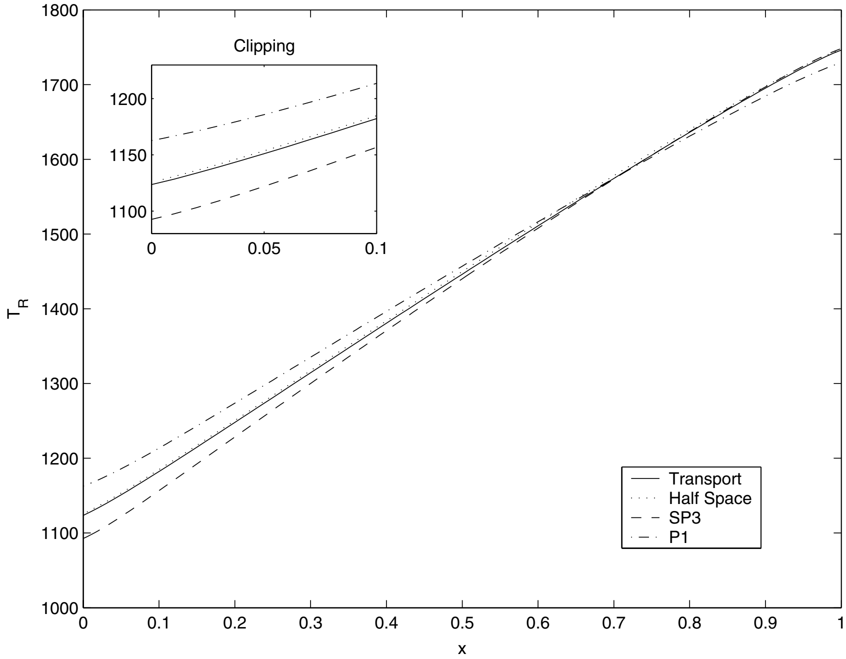


Fig. 1. Steady radiative temperature for a fixed matter temperature profile, $T(x) = 1000 + 800x$, $T_b(0) = 1000$, $T_b(1) = 1800$. Diffusive regime. Four frequency bands.

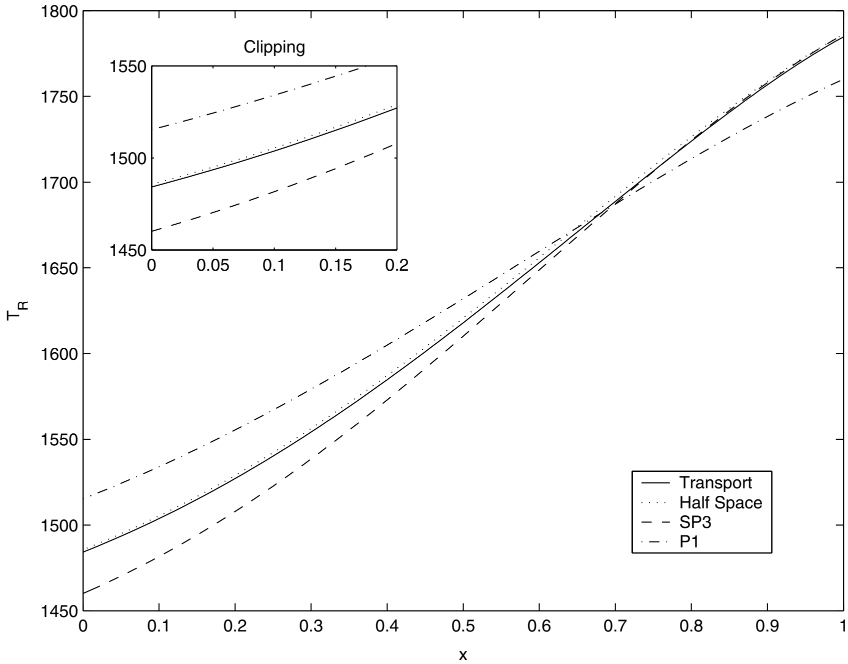


Fig. 2. Steady radiative temperature for a fixed matter temperature profile, $T(x) = 500 + 1500x$, $T_b(0) = 500$, $T_b(1) = 2000$. Transport regime. Four frequency bands.

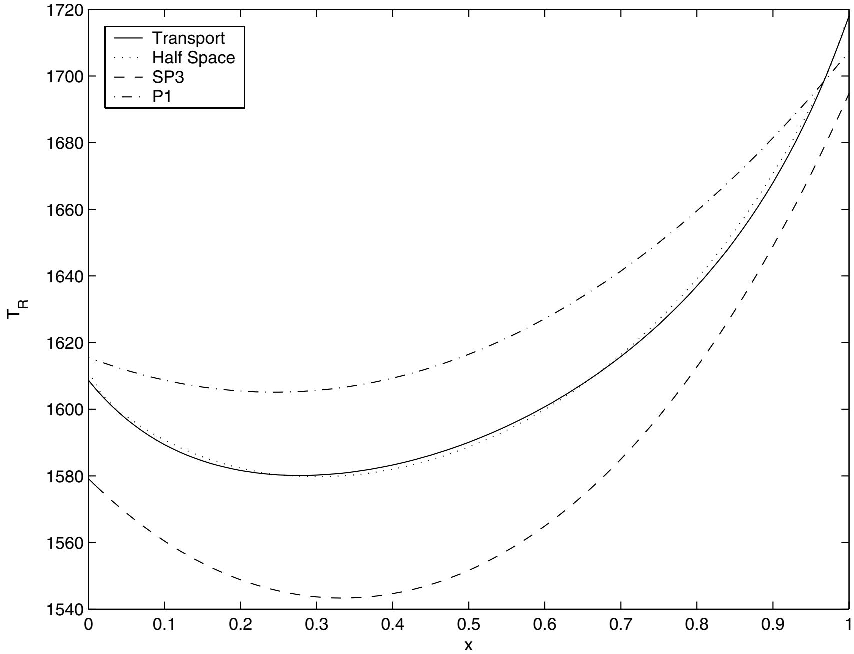


Fig. 3. Steady radiative temperature for a fixed matter temperature profile, $T(x) = 500$, $T_b(0) = 1500$, $T_b(1) = 2000$. Transport regime. Four frequency bands.

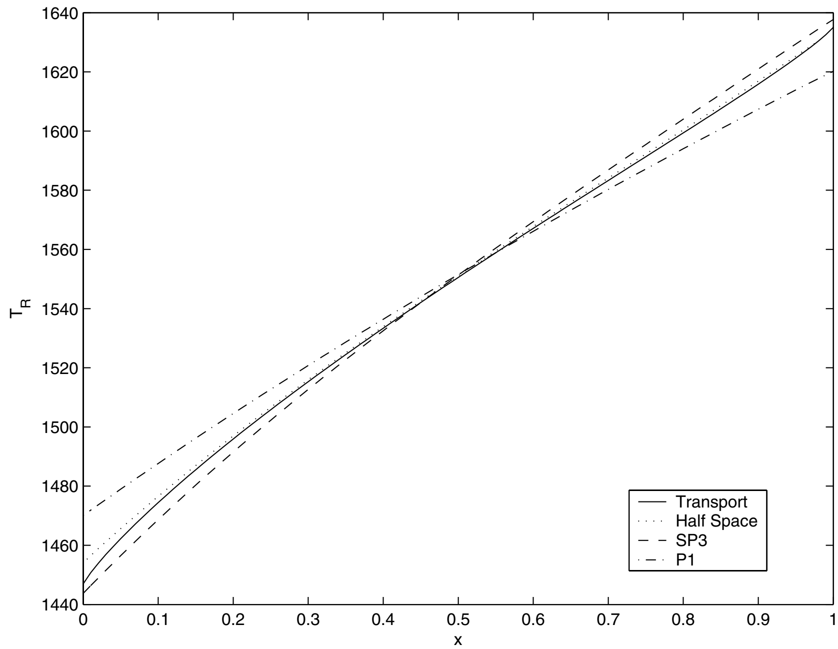


Fig. 4. Steady radiative temperature for the coupled equations, $T_b(0) = 1000$, $T_b(1) = 1800$. Transport regime. Four frequency bands.

$\kappa_2 = 0.01\ \mathrm{m}^{-1}$, $\kappa_3 = 1\ \mathrm{m}^{-1}$, $\kappa_4 = \infty$. We choose the same absorption coefficients in Fig. 3. However, while in the first two cases the boundary temperature agreed with the interior matter temperature we chose here a much higher boundary temperature which corresponds to heat flux entering the medium. The half space model is far more accurate than the diffusive approximations.

In our next test case, we consider the transport equation coupled to the heat equation. We use $k = h = 1$, $\alpha = 0$ and $\rho = 0$. The outside temperature is $T_b = 1000$ at the left and $T_b = 1800$ at the right boundary. The scattering and absorption coefficients are chosen as in our second and third uncoupled test cases. In Fig. 4, we show the steady radiative temperature. Again, the new half space model agrees best with the full transport solution.

## 4. Conclusion

- We successfully combined the half space moment approach capturing spatial anisotropy and the multigroup approach for frequency-dependent absorption and emission and obtained a very accurate, yet simple model. In our experience, the computational cost for the half space moment model is usually only a few times higher than that for diffusive models, while the solution of the full transport equation has a cost that is two orders of magnitude higher [4].
- In numerical experiments the half space moment model has shown to be valid in all physical regimes. Mathematical analyses [3,5] show that the model reproduces the diffusive limit and the free streaming limit of the transport equation. The multigroup half space moment model uses piecewise constant opacities. These means are nearly perfect near radiative equilibrium and the errors due to the averaging increase as we get farther from it. In fact, in a very transparent case we would find greater errors than in an opaque one. Another limit of our approach could become apparent in multi-dimensional situations. In one dimension in a reasonable setting the photon distribution function can only be discontinuous at the interface between left-going and right-going directions. This is perfectly taken into account for by using half space moments. In more than one dimension the location of possible discontinuities and therefore a reasonable partition of the space of directions is not known a priori. Further mathematical research on the validity of the model has to be performed.
- The grey half space moment model has already been generalized to more dimensions by using general partitions of the unit sphere. An article describing the results is in preparation. This partial space moment approach can also be applied to other closures for the moment system. Research in this direction is currently under investigation.
- Further applications of the model are currently under investigation, such as Inertial Confinement Fusion and reentry flows.

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